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ANALYTIC DENSITY IN LIE GROUPS

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ABSTRACT

A subgroup H of an analytic group G is said to be *analytically* dense if the only analytic subgroup of G containing H is G itself. The main purpose of this paper is to give sufficient conditions on G (analogous to those of [8], [9], and [7] in the case of Zariski density) which guarantee the analytic density of cofinite volume subgroups H. First we consider the case of arbitrary cofinite volume subgroups (Theorem 5 and its corollaries). Then we specialize to lattices, and prove the following result (Theorem 8): *Let G be an analytic group whose radical is simply connected and whose Levi factor has no compact part and a finite center. Then any lattice in G is analytically dense.* In proving this use is made of a result of Montgomery which also implies that for any simply connected solvable group, cocompactness of a closed subgroup implies analytic density. In the case of a solvable group with real roots this means analytic density and cocompactness are equivalent and thus completes a circle of ideas raised in Saito [13]. In Corollary 9 we deal with a related local condition. Finally in Theorem 10 and its corollaries we apply these considerations to prove a homomorphism extension theorem and an isomorphism theorem for 1-dimensional cohomology.

In Mosak and Moskowitz [7] we extended the density theorems of [8] and [9] from algebraic to analytic linear groups. In the present paper, instead of considering Zariski density, we consider the analogous but not equivalent notion of analytic density which is defined as follows: A subgroup H of an analytic group G is said to be *analytically dense* if the only analytic subgroup of G containing H is G itself. This concept was first studied (for solvable groups, under the name "full") by Mostow [10] and later by Saito [13]. In the case of simply connected nilpotent groups the fact that the notions of analytic and Zariski density coincide is a basic result of Malcev (see e.g. [11] Theorems 2.1–2.3). On the other hand, as was pointed out in [8, p. 25], if G is a solvable analytic linear group all of whose eigenvalues are real, then analytic density of H

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is equivalent to compactness of G/H, but is stronger than Zariski density of H. Our main purpose here is to give sufficient conditions on G (analogous to those of [8], [9], and [7] in the case of Zariski density) which guarantee the analytic density of cofinite volume subgroups H.

First we consider the case of arbitrary cofinite volume subgroups (Theorem 5 and its corollaries). Then we specialize to lattices, and prove the following result (Theorem 8): Let G be an analytic group whose radical R is simply connected and such that the Levi factor S has no compact factors and a finite center. Then any lattice in G is analytically dense. In proving this, use is made of a result of Montgomery [6] which also implies (see Remark 2 following Theorem 8) that for any simply connected solvable group, with real roots or not, cocompactness of a closed subgroup implies analytic density. In the case of a solvable group with real roots this means analytic density and cocompactness are equivalent and this completes a circle of ideas raised in Saito [13]. In Corollary 9 we deal with a related local condition. Finally in Theorem 10 and its corollaries we apply these considerations to prove a homomorphism extension theorem which generalizes and unifies a classical result of Malcev ([11], Chap. 2) for uniform subgroups of simply connected nilpotent groups, and a result of Saito ([13] p. 166) which was rediscovered by Gorbacevic [1], for lattices in simply connected solvable groups with real roots. In addition we prove an isomorphism theorem for 1-dimensional cohomology which generalizes our corresponding result in [7] as well as that of van Est (see [11] p. 122). In a subsequent paper we will give a number of applications of these results.

We begin with some preliminaries.

Lemma 1. *Let G be an analytic group with a semi-direct product decomposition $G = R \times_\theta S$ where R is the radical of G and S is a Levi factor. Suppose also that R is simply connected. Then a maximal compact subgroup of S is a maximal compact of G.*

Proof. Let K be a maximal compact of G, and let $\pi: G \to G/R$ be the canonical projection. Then $\pi|_K$ is 1:1, since simple connectivity of R implies that $K \cap R = (1)$ ([4], p. 138). Thus $\dim K = \dim \pi(K) \leqq \dim(\text{max compact of } S) \leqq \dim(\text{max compact of } G)$. Since maximal compacts are connected, a maximal compact of S must be a maximal compact of G.

The following lemma is well-known, but a proof seems hard to find in the literature.

Lemma 2. *Let G an analytic group with H and K analytic subgroups and K*

normal. Then $G = HK$ *if and only if* $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, *where* $\mathfrak{g}$, $\mathfrak{h}$, *and* $\mathfrak{k}$ *are the corresponding Lie algebras.*

PROOF. Suppose $G = HK$. Let $H \times K$ act on G by $(h, k)g = hgk^{-1}$. Since $\theta_{H\times K}(1) = HK = G$ this action is transitive and hence by Th. 2.5 Chap. 1 of [4], G is $H \times K$-equivariantly homeomorphic with $H \times K/\mathrm{Stab}_{H\times K}(1)$. In particular the multiplication map $H \times K \to G$ is open. Let U be a canonical neighborhood of 1 in G and V small enough so that $V^2 \subseteq U$. Let $V_H = H \cap V$ and $V_K = K \cap V$. Then these are canonical neighborhoods in H and K respectively and by the above $V_H V_K$ contains a neighborhood W of 1 in G which is canonical since $W \subseteq V^2 \subseteq U$. If $g = \exp X$ is in W then $g = hk$ where $h \in V_H$ and $k \in V_K$. Hence $\exp X = \exp Y \cdot \exp Z$ where $Y \in \mathfrak{h}$ and $Z \in \mathfrak{k}$. But the latter is

$$\exp(Y + Z + \tfrac{1}{2}[Y, Z] + \cdots) = \exp(Y + Z') \quad \text{where } Z' \in \mathfrak{k}$$

since $\mathfrak{k}$ is an ideal. By taking Y and Z small enough, $\exp(Y + Z') \in U$. It follows that $X = Y + Z'$. This proves the claim for small X. By scaling we see that $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$.

Conversely suppose $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ and $g \in U$. Then $g = \exp X$ where X is near 0. By assumption $X = Y + Z$ where $Y \in \mathfrak{h}$ and $Z \in \mathfrak{k}$ are near enough to 0 for the Campbell–Hausdorff series to converge. Accordingly

$$\exp(-Y)g = \exp(-Y)\exp(Y + Z) = \exp(Z + \tfrac{1}{2}[-Y, Y + Z] + \cdots).$$

Now $[-Y, Y + Z] = [Z, Y] \in \mathfrak{k}$ and similarly all subsequent terms are in $\mathfrak{k}$, since $\mathfrak{k}$ is an ideal. Thus $\exp(-Y)g$ equals $\exp(Z + Z')$ where $Z' \in \mathfrak{k}$. This means $g = \exp Y \cdot \exp(Z + Z') \in HK$ for each $g \in U$. Now since U generates G and K is normal, $G = HK$.

PROPOSITION 3. *Let* G *be an analytic group with* R *simply connected, and* S *having finite center. If* L *is a dense analytic subgroup of* G, *then* $L = G$.

PROOF. By Lemma 3.1 of [3] (slightly adjusted from the linear case), since S has finite center and R is simply connected, the Levi decomposition of G is semi-direct. Let K be a maximal compact subgroup of S, which by Lemma 1 is a maximal compact of G. By Goto [2, Th. 1], $G = L \cdot T$, where $T = \mathrm{rad}(K) \subseteq S$ is a torus. Thus G also equals LS. But L is normal ([4], p. 190), so by Lemma 2 we have $\mathfrak{g} = \mathfrak{l} + \mathfrak{t}$ and $\mathfrak{g} = \mathfrak{l} + \mathfrak{s}$. Hence $\mathfrak{g}/\mathfrak{l}$ is abelian, on the one hand, and semisimple, on the other. It follows that $G = L$.

REMARK. As the proof shows, the hypothesis that $Z(S)$ is finite is unnecessarily strong, and is used only to ensure that the Levi decomposition is semi-direct. Nevertheless, for convenience, we will continue to assume that $Z(S)$ is finite in the sequel.

A complex analytic subgroup of $\mathrm{Gl}(n, \mathbf{C})$ is called *reductive* if all its finite dimensional holomorphic representations are completely reducible. An alternative characterization of such a group is the following: G is reductive if and only if its Lie algebra is reductive and $Z(G)_0$, the identity component of the center, is diagonalizable.

LEMMA 4. *A normal analytic subgroup L of a complex reductive group G is itself reductive.*

PROOF. Let $\mathfrak{l}$ be the Lie algebra of L. It is easy to see that $\mathfrak{l}$ is reductive, and that $\mathfrak{z}(\mathfrak{l}) \subseteq \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. Hence $Z(L)_0 \subseteq Z(G)_0$. Since the latter is diagonalizable so is the former.

We now turn to closed subgroups of cofinite volume.

THEOREM 5. *Let G be an analytic group whose radical R is simply connected, and whose Levi factor S has a finite center and no compact factor. If* $\mathrm{Ad}(G)$ *is minimally quasibounded* (see [7]) *then any closed subgroup H of cofinite volume in G is analytically dense.*

COROLLARY 6. *Let G be a real analytic group where R is simply connected, and S has no compact part. Then closed subgroups of cofinite volume are analytically dense in G under any of the following additional hypotheses:*

(1) *G is semisimple.*

(2) *S has finite center, and G, or even* $\mathrm{Ad}(G)$, *is m.a.p.*

(3) *G is a subgroup of* $\mathrm{Gl}(V)$, *and R acts on V with real eigenvalues.*

COROLLARY 7. *Let G be a complex analytic group. Then closed subgroups H of cofinite volume in G are analytically dense in either of the following two cases:*

(1) *G has a simply connected radical.*

(2) *G is linear and reductive.*

PROOF OF THEOREM 5. Let L be an analytic subgroup of G containing H. To prove $L = G$ we may assume L is closed, for otherwise, Proposition 3 shows that we may replace L by its closure. Furthermore, since H normalizes L, the density theorem [7, Th. 3.4] shows that L is normal in G. Therefore, G/L is a compact group, and since S is m.a.p., S maps trivially into G/L. Thus $S \subseteq L$, so $G = RS = RL$; hence $G/L = R/R \cap L$, and G/L must be a torus. On the other hand, as in the proof of Proposition 3, $G = R \times_\theta S$. Since L contains S, this implies that L is the semi-direct product $(R \cap L) \times_\theta S$, so $R \cap L$ is connected. Since R is simply connected, so is $R/R \cap L = G/L$. Therefore $G = L$.

PROOF OF COROLLARY 6. By the remark following Proposition 3, the hypothesis that $Z(S)$ is finite may be replaced by the hypothesis that G has a semi-direct product Levi decomposition. This proves (1). For (2), if $\mathrm{Ad}(G)$ is m.a.p. then it certainly has no rational homomorphisms into compact groups, and is therefore minimally quasibounded [7, Th. 2.5]. Now $\mathrm{Ad}(G) = \mathrm{Ad}(R)\mathrm{Ad}(S)$ is the Levi decomposition of $\mathrm{Ad}(G)$, and the hypothesis implies that $\mathrm{Ad}(S)$ has no compact part. Since $\mathrm{Ad}(S) = S/S \cap Z(G)$ and $S \cap Z(G)$ is a discrete central subgroup of S, this means S also has no compact factors. Thus (2) follows from Theorem 5. For (3), [7, Prop. 2.10] implies that G is minimally quasibounded, hence so is $\mathrm{Ad}(G)$. Also, R is simply connected by [8], and $Z(S)$ is finite. Hence (3) also follows from Theorem 5.

PROOF OF COROLLARY 7. For 1, by [4, Ch. 18, Th. 4.6], G is linear, and $\mathrm{Ad}(G)$ is minimally quasibounded [7, 2.10]; also S has no compact factors. Therefore (1) follows from Theorem 5. For (2), let L be an analytic subgroup of G containing H. Arguing as in the proof of Theorem 5, we see that L is a normal analytic subgroup of G and as such is itself reductive by Lemma 4. But a reductive complex analytic linear group is algebraic/**R** and in particular is Euclidean closed. Therefore G/L is compact. Since G is algebraic and L is a normal algebraic subgroup, G/L has a faithful C-rational linear representation (see [5]). In particular G/L is a complex analytic linear group. Since it is compact it is a point.

Our previous results concerned arbitrary closed cofinite volume subgroups. In the case of lattices certain strengthenings can be made.

THEOREM 8. *Let G be an analytic group whose radical R is simply connected and whose Levi factor S has finite center and no compact part. Then any lattice Γ is analytically dense in G.*

PROOF. Let L be an analytic subgroup containing Γ. By Proposition 3 we may assume that L is closed. By a result of H. C. Wang (see Cor. 8.27 of [11]) $R \cap \Gamma$ is a lattice in R and $\pi(\Gamma)$ is a lattice in G/R where π: $G \to G/R$ is the canonical map. From this it follows that $R/R \cap L$ is compact and $\pi(\Gamma) \subseteq \pi(L) \subseteq G/R$. Since $\pi(L)$ is connected it equals G/R by the semisimple case (Corollary 6) so $G = LR$. But then $G/L = LR/L = R/R \cap L$, a compact manifold. Now let π_S be the restriction of π to S. Then π_S maps S onto G/R

($= LR/R = L/L \cap R$ since R is normal) while $\operatorname{Ker} \pi_S = R \cap S$, a discrete central subgroup of S. This means π_S is a covering map and S is locally isomorphic to $L/L \cap R$. Let $\mathfrak{s}$, $\mathfrak{l}$, and $\mathfrak{r}$ be the respective Lie algebras. Then $\mathfrak{s} \cong \mathfrak{l}/\mathfrak{l} \cap \mathfrak{r}$. since $\mathfrak{l} \cap \mathfrak{r}$ is a solvable ideal in $\mathfrak{l}$ and $\mathfrak{l}/\mathfrak{l} \cap \mathfrak{r}$ is semisimple, the exact sequence

$$(0) \to \mathfrak{l} \cap \mathfrak{r} \to \mathfrak{l} \to \mathfrak{l}/\mathfrak{l} \cap \mathfrak{r} \to (0)$$

splits by the Levi theorem. Hence $\mathfrak{l}$ has a subalgebra $\mathfrak{s}' \cong \mathfrak{s}$; $\mathfrak{s}'$ is a Levi factor of $\mathfrak{g}$ because it is semisimple and of maximal dimension. This means $L \supseteq S'$, a Levi factor of G. Let K be a maximal compact subgroup of S'. By Lemma 1 K is a maximal compact subgroup of G; also G/L is compact, and L is connected. Thus by Montgomery [6], $G = KL = L$ (since $L \supseteq S' \supseteq K$).

Remarks. (1) For the proof of Theorem 8 it is clearly necessary to know that the assertion $G = KL$ of [6] is actually valid for an *arbitrary* maximal compact K of G. This follows from Theorem A of [6], which states that if G has a compact homogeneous space $X = G/L$, with connected stability group L, then some (maximal) compact subgroup K_0 of G acts transitively on X. For then clearly any maximal compact $K = gK_0g^{-1}$ acts transitively on X, so $G = KL$.

(2) Since a simply connected solvable group has no nontrivial compact subgroups, the result of [6] implies that if G is any simply connected solvable group and H is a closed, cocompact subgroup then H is analytically dense.

In both Theorems 5 and 8 the conditions that R be simply connected and S have no compact factors are clearly necessary. It is not obvious if the same may be said of the hypothesis that the Levi decomposition splits. This condition, in the somewhat stronger form that $Z(S)$ is finite, has already been encountered in related problems in, for example, [3]. These hypotheses will pose no problem in the applications we envisage, as they will be to subgroups of $\mathrm{Gl}(\mathfrak{g})$.

Next we consider a notion related to, but distinct from analytic density.

Definition 9. Let G be a real or complex analytic group, and H be a subgroup. We shall say that H is *locally dense* if

$$\mathfrak{l}(H) = \mathrm{lin.\,sp.}_k \{X \in \mathfrak{g} \colon \exp_G X \in H\}$$

equals $\mathfrak{g}$. Here $k = \mathbf{R}$ or $\mathbf{C}$ respectively.

For a closed subgroup H, even with cofinite volume, local density need not imply analytic density, as is shown by the example, G a compact group and H a point.

COROLLARY 9. *Let G be a solvable analytic group with real roots, or* $\mathrm{Gl}(n, \mathbf{C})$. *If H is a closed subgroup with G/H of finite volume, then H is locally dense.*

PROOF. In the solvable case by a simple covering group argument we may assume that G is simply connected. This implies that G is a linear group. Next we observe that in either case $\mathfrak{l} = \mathfrak{l}(H)$ is an ideal in $\mathfrak{g}$. For if $\exp X \in H$ and $h \in H$ then $\exp \mathrm{Ad}_h(X) = h \exp X h^{-1} \in H$. Thus $\mathfrak{l}$ is $\mathrm{Ad}(H)$-stable. By the density theorem [7] $\mathfrak{l}$ is $\mathrm{Ad}(G)$-stable, so taking infinitesimal generators shows that $\mathfrak{l}$ is $\mathrm{ad}(\mathfrak{g})$-stable and so an ideal. Let L be the corresponding (normal) analytic subgroup. Now $\exp_G$ is surjective. In the solvable case this is a result of Saito [12]. In the case of $\mathrm{Gl}(n, \mathbf{C})$ this is well known. In both cases, therefore, H is contained in the range of the exponential map, so $H \subseteq L$. Since H is analytically dense by Remark 2 above or Corollary 7, respectively, $L = G$. Therefore $\mathfrak{l} = \mathfrak{g}$.

We remark that by the same argument Corollary 9 also holds for any Lie group covered by $\mathrm{Gl}(n, \mathbf{C})$.

THEOREM 10. *Let G and G^* be simply connected solvable groups with real roots, and let H be a closed cocompact subgroup of G. Then each smooth homomorphism $\varphi: H \to G^*$ extends to a unique smooth homomorphism $\Phi: G \to G^*$.*

PROOF OF UNIQUENESS. Let Φ and Φ' be two such extensions and $d\Phi$ and $d\Phi'$ be the corresponding differentials mapping $\mathfrak{g} \to \mathfrak{g}^*$. Since $\Phi \exp = \exp d\Phi$, $\Phi' \exp = \exp d\Phi'$, and exp is bijective, $d\Phi$ and $d\Phi'$ agree on $\log(H)$. By Corollary 9, $\log(H)$ generates $\mathfrak{g}$, so $\Phi = \Phi'$.

PROOF OF EXISTENCE. Let π and π^* denote the projections from $G \times G^*$ onto G and G^*, respectively. Now $G \times G^*$ is a simply connected solvable group with real roots. Since φ is continuous its graph is a closed subgroup $G \times G^*$ (and H is isomorphic with graph φ under the map $h \to (h, \varphi(h))$. Let L be the analytic hull of graph φ in $G \times G^*$ (see [11]) and $\psi = \pi|_L$. Then ψ is a smooth homomorphism $L \to G$ so that $\psi(L)$ is an analytic subgroup of G. But $\psi(L)$ clearly contains H. Since H is analytically dense in G by Remark 2 above, $\psi(L) = G$. Now ψ induces a diffeomorphism $L/\psi^{-1}(H) \to G/H$. If we knew that $\psi^{-1}(H) = \text{graph}\,\varphi$ then $\dim(L) - \dim(\text{graph}\,\varphi) = \dim(G) - \dim(H)$. But then $\dim(L) = \dim(G)$ so that $\mathrm{Ker}\,\psi$ would be discrete and ψ a covering map. Since G is simply connected this means ψ is an isomorphism. Now let $\Phi = \pi^*|_L \cdot \psi^{-1}: G \to G^*$. Then Φ is a smooth homomorphism and $\Phi|_H = \varphi$.

To see that $\psi^{-1}(H) = \text{graph}\,\varphi$, suppose $(g, g^*) \in L$ and $\psi(g, g^*) = g \in H$. Then we show $g^* = \varphi(g)$. Since the exponential map for $G \times G^*$ is given by $(\exp_G, \exp_{G^*})$ we have $(g, g^*) = (\exp X, \exp X^*) \in L$ where $X \in \mathfrak{g}$ and $X^* \in \mathfrak{g}^*$. But L is the smallest analytic subgroup of $G \times G^*$ containing graph φ, therefore by Proposition 6 of Saito [13] we have $(X, X^*) = \Sigma\, t_i(Y_i, Y_i^*)$ where $t_i \in \mathbf{R}$, $Y_i \in \mathfrak{g}$, $Y_i^* \in \mathfrak{g}^*$, and $\varphi(\exp Y_i) = \exp Y_i^*$ for $i = 1, \dots, n$. Therefore $\varphi'(Y_i) = Y_i^*$ for all i. Since $X = \Sigma\, t_i Y_i$ and $X^* = \Sigma\, t_i Y_i^*$ we have $\varphi'(X) = \Sigma\, t_i \varphi'(Y_i) = \Sigma\, t_i Y_i^* = X^*$. This means $\varphi(g) = g^*$.

Corollary 11. *Let G be a simply connected solvable group with real roots, and let H be a closed subgroup with G/H compact. Then any automorphism α of H extends uniquely to an automorphism β of G.*

Proof. α extends to a unique smooth homomorphism β by Theorem 10. Now $\beta(G) \supseteq \beta(H) = H$ so $\beta(G)$ is an analytic subgroup of G containing H. Since H is analytically dense in G by Remark 2, $\beta(G) = G$. So $\text{Ker}\,\beta$ is a discrete central subgroup of G and β is a covering map. Since G is simply connected, β must be an automorphism of G.

As a final application of theorem 10 we prove a generalization of part of a result in [7], namely Proposition 3.2 in the case of 1-cocycles. In what follows V is a real vector space, G is a subgroup of $\text{Gl}(V)$, and cocyles are taken with respect to the natural action.

Corollary 12. *Let G be a simply connected solvable subgroup of $\text{Gl}(V)$ with real roots and H be a closed cocompact subgroup. Then each smooth 1-cocycle $\varphi: H \to V$ extends to a unique smooth 1-cocycle of G with values in V. In other words, the restriction map $H^1(G, V) \to H^1(H, V)$ is an isomorphism.*

Proof. Consider the simply connected solvable group $G \times_{\text{id}} V$ and its Lie algebra $\mathfrak{g} +_{\text{id}} V$. An easy calculation shows that $\text{Ad}_{(g,v)}(X, w) = (\text{Ad}_g(X), (I - \text{Ad}_g(X))v + g \cdot w)$. In particular this implies that $G \times_{\text{id}} V$ has only real roots. Now the map $h \to (h, \varphi(h))$ is a smooth homomorphism of $H \to G \times_{\text{id}} V$ which extends by Theorem 10 to a unique smooth homomorphism $\Phi: G \to G \times_{\text{id}} V$. Let π_G and π_V be the projections on G and V respectively. Then $\pi_G \cdot \Phi$ is a smooth homomorphism from $G \to G$ whose restriction to H is the identity and so by Theorem 10 is the identity on G. Let $\psi = \pi_V \cdot \Phi$; then $\psi|_H = \varphi$ and since $\Phi(g) = (g, \psi(g))$ and Φ is a homomorphism, ψ is a 1-cocycle. Because Φ is unique so is ψ.

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